The variance of a discrete uniform distribution is $\sigma^2 = (n^2 - 1)/12$ provided x starts at 1 and increases by 1 to n.

PROOF:

$$\begin{aligned}\sigma^2 &= \Sigma(x-\mu)^2 \tfrac{1}{n} \\ &= \tfrac{1}{n}\Sigma(x-\mu)(x-\mu) \\ &= \tfrac{1}{n}\Sigma(x^2 - 2\mu x + \mu^2) \\ &= \tfrac{1}{n}\left(\Sigma x^2 - 2\mu\Sigma x + n\mu^2\right) \\ &= \tfrac{1}{n}\left(\Sigma x^2 - 2\mu \tfrac{n}{n}\Sigma x + n\mu^2\right) \\ &= \tfrac{1}{n}\left(\Sigma x^2 - 2n\mu^2 + n\mu^2\right)\end{aligned}$$

So,

$$\sigma^2 = \frac{1}{n}\Sigma x^2 - \mu^2 \tag{1}$$

Proposition:

$$\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

To prove this proposition, we use induction. To do this, first show it holds for $n = 1$:

$$\sum_{x=1}^{1} x^2 = 1^2 = 1 \qquad \text{and} \qquad \frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1(2)(3)}{6} = 1$$

Now assume (2) holds to show it holds for $n + 1$:

$$\begin{aligned}\sum_{x=1}^{n} x^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \\ &= \frac{(n+1)(2n^2+n) + 6(n+1)^2}{6} \\ &= \frac{(n+1)[(2n^2+n) + 6(n+1)]}{6} \\ &= \frac{(n+1)[2n^2+n+6n+6]}{6} \\ &= \frac{(n+1)[2n^2+7n+6]}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n+1)(n+2)(2n+2+1)}{6}\end{aligned}$$

Thus, (2) holds because the above simplifies into the following:

$$\sum_{x=1}^{n+1} x^2 = \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}.$$

Substituting (2) and $\mu = (n + 1)/2$ into (1) yields:

$$\begin{aligned}
\sigma^2 &= \frac{1}{n}\frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{n(n+1)(2n+1)}{6n} - \frac{(n+1)^2}{4} \\
&= \frac{n(2n^2+3n+1)}{6n} - \frac{n^2+2n+1}{4} \\
&= \frac{2n^3+3n^2+n}{6n} - \frac{n^2+2n+1}{4} \\
&= \frac{2}{2}\frac{2n^3+3n^2+n}{6n} - \frac{3n}{3n}\frac{n^2+2n+1}{4} \\
&= \frac{4n^3+6n^2+2n}{12n} - \frac{3n^3+6n^2+3n}{12n} \\
&= \frac{4n^3+6n^2+2n-3n^3-6n^2-3n}{12n} \\
&= \frac{n^3-n}{12n} \\
&= \frac{n^2-1}{12}
\end{aligned}$$

The variance of a continuous uniform distribution is $\sigma^2 = (b-a)^2/12$ where x is between a and b.

PROOF:

$$
\begin{aligned}
\sigma^2 &= \int_a^b (x-\mu)^2 \tfrac{1}{b-1}\,dx \\
&= \tfrac{1}{b-a}\int_a^b (x-\mu)(x-\mu)dx \\
&= \tfrac{1}{b-a}\int_a^b (x^2-2\mu x+\mu^2)dx \\
&= \tfrac{1}{b-a}\left[\tfrac{1}{3}x^3-2\mu\tfrac{1}{2}x^2+\mu^2 x+c\right]_a^b \\
&= \tfrac{1}{b-a}\left[\tfrac{1}{3}(b^3-a^3)-\mu(b^2-a^2)+\mu^2(b-a)\right] \\
&= \tfrac{1}{b-a}\left[\tfrac{1}{3}(b^3-a^3)-\tfrac{1}{2}(b+a)(b^2-a^2)+\tfrac{1}{4}(b+a)^2(b-a)\right] \\
&= \tfrac{1}{12(b-a)}\left[4(b-a)(b^2+ab+a^2)-6(b+a)(b^2-a^2)+3(b+a)^2(b-a)\right] \\
&= \tfrac{1}{12(b-a)}\left[4(b-a)(b^2+ab+a^2)-6(b+a)(b-a)(b+a)+3(b+a)^2(b-a)\right] \\
&= \tfrac{1}{12}\left[4(b^2+ab+a^2)-6(b+a)^2+3(b+a)^2\right] \\
&= \tfrac{1}{12}\left[4(b^2+ab+a^2)-3(b+a)^2\right] \\
&= \tfrac{1}{12}\left[4b^2+4ab+4a^2-3(b^2+2ab+a^2)\right] \\
&= \tfrac{1}{12}\left[4b^2+4ab+4a^2-3b^2-6ab-3a^2\right] \\
&= \tfrac{1}{12}\left[b^2-2ab+a^2\right] \\
&= \tfrac{(b-a)^2}{12}
\end{aligned}
$$